

August 2008

Doubly special relativity and translation invariance

S. Mignemi[‡]

Dipartimento di Matematica, Università di Cagliari
viale Merello 92, 09123 Cagliari, Italy
and INFN, Sezione di Cagliari

Abstract

We propose a new interpretation of doubly special relativity (DSR) based on the distinction between the momentum and the translation generators in its phase space realization. We also argue that the implementation of DSR theories does not necessarily require a deformation of the Lorentz symmetry, but only of the translation invariance.

P.A.C.S. Numbers: 11.30.Cp, 03.30.+p

[‡] e-mail: smignemi@unica.it

Introduction

In recent years, the idea that special relativity should be modified for energies close to the Planck scale κ , in such a way that κ becomes an observer-independent parameter of the theory like the speed of light, has been extensively debated [1-4]. This hypothesis is motivated by the consideration that the Planck energy sets a limit above which quantum gravity effects become important, and its value should therefore not depend on the specific observer, as would be the case in special relativity. Of course, this postulate must be implemented in such a way that the principle of relativity, i.e. the equivalence of all inertial observers, be still valid. The theory based on these assumptions has been named doubly special relativity (DSR) [1].

DSR models are realized by deforming the Poincaré invariance of special relativity. Their main physical consequences are the modification of the dispersion relations of elementary particles and the existence of a nonlinear addition law for the momenta. In particular, one is lead to identify κ with a maximal value of the energy or the momentum for elementary particles.

These nontrivial effects have been used to derive experimentally verifiable predictions, for example to explain the observed threshold anomalies in ultrahigh-energy cosmic rays [5]. Another natural, although not necessary, consequence of the formalism is the noncommutativity of the spacetime geometry. In particular, DSR fits very well in the formalism of κ -Poincaré algebras [6], that postulates a quantum deformation of the Poincaré group acting on noncommutative spacetime [4], although the two theories cannot be considered equivalent [7]. One drawback of DSR is however that the deformation of special relativity resulting from its postulates is not unique, and several inequivalent models can be constructed.

DSR is usually associated with the deformation of the Lorentz symmetry. In this paper we wish to point out that its really distinguishing feature is not the deformation of the Lorentz symmetry, but rather that of the translation symmetry. As mentioned before, the main phenomenological consequences of DSR are a deformation of the addition law of momenta and of the dispersion law of the elementary particles. These clearly depend only on the nontrivial action of translations generators on momenta in phase space. In fact, the

deformed dispersion relation is given by the Casimir invariant of the translations.

The lack of the necessity of deforming the Lorentz invariance is clearly illustrated for example by the Snyder realization of DSR [8,9]. The Snyder model was originally proposed [8] in order to show the possibility of introducing a noncommutative spacetime without breaking the Lorentz symmetry. It was then observed that it can be interpreted as a DSR model [9], but the physical implications of this fact were not further investigated. In the original formulation of the model [8], the Poincaré invariance is realized canonically, but, as we shall show, it is possible to give an alternative interpretation in terms of DSR.

Another important point we wish to stress is that, from an algebraic point of view, the realization of DSR does not necessarily require a deformation of the Poincaré group, as in the κ -Poincaré formalism, but can be carried out in a classical framework through a nonlinear action of the Poincaré group on phase space. This interpretation of DSR is close in the spirit to the proposal of [3] and has been stressed especially in [10]. In essence, at the classical level one can deform the generators of the Poincaré group in such a way that obey the standard Poincaré algebra but nevertheless act nontrivially on phase space variables.

In the present paper we show that these ideas can be implemented in a natural way if, in analogy with what happens in curved space, one distinguishes the translation generators from the canonical momenta. This observation also clarifies the physical origin of the composition law of momenta proposed in ref. [11] and usually adopted in the DSR literature, whose interpretation was rather obscure. In fact, in that paper the addition of momenta was obtained through the introduction of unphysical auxiliary variables, that in our interpretation are identified with the generators of the translation symmetry.

To illustrate these considerations, we discuss the Snyder model from a DSR point of view. The same formalism can of course be applied to more traditional DSR models, where also the action of the Lorentz group is deformed.

The model

Let us start by considering the classical action of the Poincaré algebra on the phase

space of special relativity¹. The Poincaré algebra is spanned by the Lorentz generators $J_{\mu\nu}$ and the translation generators T_μ , obeying Poisson brackets

$$\begin{aligned}\{J_{\mu\nu}, J_{\rho\sigma}\} &= \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho}, \\ \{J_{\mu\nu}, T_\lambda\} &= \eta_{\mu\lambda}T_\nu - \eta_{\nu\lambda}T_\mu, \quad \{T_\mu, T_\nu\} = 0.\end{aligned}\tag{1}$$

Its realization in canonical phase space, with Poisson brackets

$$\{x_\mu, x_\nu\} = \{p_\mu, p_\nu\} = 0, \quad \{x_\mu, p_\nu\} = \eta_{\mu\nu},\tag{2}$$

is obtained through the identification

$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad T_\mu = p_\mu,\tag{3}$$

which yields the transformation laws for the phase space coordinates,

$$\begin{aligned}\{J_{\mu\nu}, x_\lambda\} &= \eta_{\mu\lambda}x_\nu - \eta_{\nu\lambda}x_\mu, & \{J_{\mu\nu}, p_\lambda\} &= \eta_{\mu\lambda}p_\nu - \eta_{\nu\lambda}p_\mu, \\ \{T_\mu, x_\nu\} &= \eta_{\mu\nu}, & \{T_\mu, p_\nu\} &= 0.\end{aligned}\tag{4}$$

This formalism can be easily generalized to de Sitter space with cosmological constant Λ . The de Sitter algebra is given by

$$\begin{aligned}\{J_{\mu\nu}, J_{\rho\sigma}\} &= \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho}, \\ \{J_{\mu\nu}, T_\lambda\} &= \eta_{\mu\lambda}T_\nu - \eta_{\nu\lambda}T_\mu, & \{T_\mu, T_\nu\} &= -\Lambda J_{\mu\nu}.\end{aligned}\tag{5}$$

Usually, the algebra is realized in terms of the isometries of a hyperboloid of equation $\xi_A^2 = -1/\Lambda$, embedded in flat five-dimensional space with metric $\eta_{AB} = \text{diag}(1, -1, -1, -1, -1)$. In the following, we shall denote π_A the momenta canonically conjugate to ξ_A . The Lorentz generators $J_{\mu\nu}$ are identified with the corresponding generators of the five-dimensional algebra, while the translation generators T_μ are identified with $\sqrt{\Lambda} J_{4\mu}$.

The realization of the de Sitter algebra in four-dimensional phase space, with Poisson brackets (2), depends on the specific coordinates chosen on the hyperboloid. In general, the

¹ We adopt the notations $\mu = 0, \dots, 3$, $A = 0, \dots, 4$, $i = 1, \dots, 3$, $x \cdot p \equiv x_\mu p_\mu$.

Lorentz generators maintain the canonical form, $J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$, while the form of the translation generators depends on the parametrization of the hyperboloid. A convenient choice is given by Beltrami (projective) coordinates [12,13], $x_\mu = \xi_\mu / \sqrt{\Lambda} \xi_4$, with canonically conjugate momentum $p_\mu = \sqrt{\Lambda} \xi_4 \pi_\mu$, in terms of which the translation generators read

$$T_\mu = p_\mu - \Lambda x \cdot p x_\mu. \quad (6)$$

It is then evident that in de Sitter spacetime the generators of the translation symmetry cannot be identified with the canonical momenta p_μ .

Under translations, the Beltrami coordinates x_μ and p_μ transform as

$$\{T_\mu, x_\nu\} = -\eta_{\mu\nu} + \Lambda x_\mu x_\nu, \quad \{T_\mu, p_\nu\} = -\Lambda(x \cdot p \eta_{\mu\nu} + x_\mu p_\nu). \quad (7)$$

Thus the conserved quantity is not the canonical momentum p_μ , as should be obvious since the hamiltonian of a free particle in de Sitter spacetime is position-dependent², but the quantity associated with the translation generator, given by (6). The rules for the composition of momenta are dictated by the conservation of T_μ and not of p_μ .

We pass now to consider the case of DSR. As discussed previously, we adopt the point of view that the symmetry algebra maintains its classical form, but its action on phase space is nonlinear. Since our aim is to show the relevance of translation invariance, we consider the specific example of the Snyder model, whose most noticeable feature is that the Lorentz invariance is realized linearly in the standard way, but our considerations can be extended to any other DSR model.

As it was shown in ref. [11], the transformation laws of any DSR model can be obtained by defining the physical momentum p_μ in terms of auxiliary variables $P_\mu = U(p_\mu)$ that satisfy canonical transformation laws. The deformed dispersion relation is then given by writing the classical relation for the auxiliary variables, $P^2 = m^2$ in terms of the physical variables p_μ . Also the addition law of momenta is obtained by pulling back to the physical momenta p_μ the classical law for the variables P_μ .

² For example, in Beltrami coordinates the hamiltonian of a free particle is given by $H = \frac{1}{2} (1 - \Lambda x^2)[p^2 - \Lambda(x \cdot p)^2]$.

We propose that, in analogy with the case of de Sitter space, the generators of translations T_μ should not be identified with the momenta p_μ , but rather with the auxiliary variables P_μ . This choice clarifies the physical significance of the auxiliary variables and of the addition law for momenta proposed in [11].

In particular, in the case of the Snyder model, one chooses [13]

$$P_\mu = U(p_\mu) = \frac{p_\mu}{\sqrt{1 - \Omega p^2}}, \quad (8)$$

with inverse

$$p_\mu = U^{-1}(P_\mu) = \frac{P_\mu}{\sqrt{1 + \Omega P^2}}. \quad (9)$$

In a DSR interpretation, the Snyder model can then be characterized by the explicitly Lorentz invariant deformed dispersion relation $p^2/(1 - \Omega p^2) = m^2$, i.e. $p^2 = m^2/(1 + \Omega m^2)$, where $\Omega = 1/\kappa^2$ is the Planck area. In this form, the dispersion relation looks like a redefinition of the mass (notice that the dispersion relation for massless particles maintain its classical form), however some nontrivial consequences follow^{3,4}.

For example, for $\Omega > 0$ the model admits a maximal mass κ , and is similar to other DSR models, which admit a maximum value for the momentum or the energy of a particle. For $\Omega < 0$, instead, there is no limit value for the mass. In the quantum theory however emerges the existence of a minimal value for the momentum, as in the similar model discussed in [14]. In the following, we consider the case of positive Ω .

From the structure of (8) it follows that the action of the Lorentz group on the momentum variables is not affected, while that of translations is deformed. This illustrates the fact that the most relevant characteristic for the implementation of DSR is the deformation of the action of translations (and hence a modified composition law of momenta) and not that of Lorentz transformations, as usually postulated.

³ The original paper [8] gives a different interpretation of the physics, in which the classical dispersion relation $p^2 = m^2$ is maintained.

⁴ It may be interesting to notice that the Snyder model can be derived from a 5-dimensional momentum space of coordinates π_A , constrained by $\pi_A^2 = -1/\Omega$, in a way dual to that used for de Sitter spacetime [9].

In order to realize the model in spacetime, it is natural to introduce position variables x_μ that transform covariantly with respect to the momenta. These can be defined as [13,15]

$$x_\mu = \sqrt{1 + \Omega P^2} X_\mu, \quad (10)$$

where X_μ are the variables canonically conjugate to the P_μ .

With this definition, the Poisson brackets between the new phase space coordinates are no longer canonical, and the position space becomes noncommutative, realizing the proposal of Snyder [8],

$$\{x_\mu, x_\nu\} = -\Omega(x_\mu p_\nu - x_\nu p_\mu), \quad \{p_\mu, p_\nu\} = 0, \quad \{x_\mu, p_\nu\} = \eta_{\mu\nu} - \Omega p_\mu p_\nu. \quad (11)$$

In terms of the physical coordinates x_μ and p_μ , the generators of the Poincaré group read

$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad T_\mu = P_\mu = p_\mu / \sqrt{1 - \Omega p^2}. \quad (12)$$

The transformation laws of x_μ and p_μ under the Lorentz subalgebra maintain the canonical form, while under translations become

$$\{T_\mu, x_\nu\} = \frac{\eta_{\mu\nu}}{\sqrt{1 - \Omega p^2}}, \quad \{T_\mu, p_\nu\} = 0. \quad (13)$$

Therefore, the effect of the translations on the position coordinates becomes momentum dependent and increases for near Planck-mass particles.

The sum of the momenta of two particles with momenta $p_\mu^{(1)}$ and $p_\mu^{(2)}$ is given in general by [3]

$$p_\mu^{(12)} = U^{-1}[U(p_\mu^{(1)}) + U(p_\mu^{(2)})]$$

and in our case it can be readily obtained from (8) and (9),

$$p_\mu^{(12)} = \frac{\sqrt{1 - \Omega(p_\mu^{(2)})^2} p_\mu^{(1)} + \sqrt{1 - \Omega(p_\mu^{(1)})^2} p_\mu^{(2)}}{\sqrt{1 - \Omega^2(p_\mu^{(1)})^2(p_\mu^{(2)})^2 + 2\Omega p_\mu^{(1)} \cdot p_\mu^{(2)} \sqrt{(1 - \Omega(p_\mu^{(1)})^2)(1 - \Omega(p_\mu^{(2)})^2)}}}. \quad (14)$$

Notice that this expression is nontrivial even for massless particles. Using (14) one may calculate the effect of the deformed transformations on the scattering of ultra-high-energy

cosmic rays by the cosmic background radiation [5]. We shall not perform the calculation in detail, but a correction of the classical threshold of electron production arises as in the other DSR models.

It is also possible to define a dynamics for the free particle, by introducing a hamiltonian. This can be obtained simply substituting (8) into the classical hamiltonian,

$$H = \frac{P^2}{2} = \frac{1}{2} \frac{p^2}{1 - \Omega p^2}. \quad (15)$$

The Hamilton equations can then be derived taking into account the symplectic structure (11). They read

$$\dot{x}_\mu = \frac{p_\mu}{1 - \Omega p^2}, \quad \dot{p}_\mu = 0. \quad (16)$$

The 3-velocity of a free particle, defined as $\mathbf{v}_i = \dot{x}_i/\dot{x}_0$ maintains its classical expression p_i/p_0 and cannot exceed the speed of light. The same expression is obtained by the alternative definition $\mathbf{v}_i = \partial p_0/\partial p_i$.

Also a natural definition of the spacetime metric can be given [13], yielding $ds^2 = (1 - \Omega p^2)dx^2$. As usual in DSR, the metric depends explicitly on the momentum [15].

Conclusion

We have shown that DSR can be interpreted as a classical relativistic mechanic model with nontrivial generators of translations, and have discussed this point in the special case of the Snyder model. Similar considerations can be applied to other DSR models. For example, the results of [10] for the Maguejo-Smolín model [3] can be interpreted in this perspective. The discussion of this specific model, which is Lorentz invariant, shows also that the implementation of DSR does not necessarily require in general a deformation of the Lorentz symmetry, but only of the translational one, contrary to the common view. More general Lorentz-invariant DSR models can also be obtained starting from different Lorentz-invariant deformations of the dispersion relation of elementary particles, of the form $f(p^2) = m^2$.

Of course the crucial point for the physical interpretation is that the physically measured variables should be identified with the canonical momenta. This would require an

operational definition of momentum measurements in DSR that to our knowledge is still lacking.

We notice that our interpretation is not intended to solve the problems of DSR such as the so-called soccer ball problem (i.e. the fact that if DSR had to hold also for macroscopical objects, their momentum should not exceed the Planck scale), but just to give a clearer interpretation of the formalism in classical terms, alternative for example to the quantum-group based κ -Poincaré formalism. From the discussion above, it should also be evident that DSR is no more equivalent to special relativity than de Sitter space is to flat space.

References

- [1] G. Amelino-Camelia, Phys. Lett. **B510**, 255 (2001), Int. J. Mod. Phys. **D11**, 35 (2002).
- [2] J. Kowalski-Glikman, Phys. Lett. **A286**, 391 (2001).
- [3] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002); Phys. Rev. **D67**, 044017 (2003).
- [4] For a review, see J. Kowalski-Glikman, Lecture Notes in Physics **669**, 131 (2005).
- [5] G. Amelino-Camelia and T. Piran, Phys. Lett. **B497**, 265 (2001); Phys. Rev. **D64**, 036005 (2001).
- [6] S. Majid and H. Ruegg, Phys. Lett. **B334**, 348 (1994); J. Lukierski, H. Ruegg and W.J. Zakrzewski, Ann. Phys. **243**, 90 (1995).
- [7] G. Amelino-Camelia, J. Kowalski-Glikman, G. Mandanaci and A. Procaccini, Int. J. Mod. Phys. **A20**, 6007 (2005).
- [8] H.S. Snyder, Phys. Rev. **71**, 38 (1947).
- [9] J. Kowalski-Glikman and S. Nowak, Int. J. Mod. Phys. **D13**, 299 (2003).
- [10] S. Ghosh and P. Pal, Phys. Rev. **D75**, 105021 (2007).
- [11] S. Judes and M. Visser, Phys. Rev. **D68**, 045001 (2003).

- [12] H.Y. Guo, C.G. Huang, Z. Xu and B. Zhou, Phys. Lett. **A331**, 1 (2004).
- [13] S. Mignemi, [arXiv:0802.1129](#).
- [14] A. Kempf, G. Mangano and R.B. Mann, Phys. Rev. **D52**, 1108 (1995).
- [15] S. Mignemi, Phys. Rev. **D68**, 065029 (2005).
- [16] J. Magueijo and L. Smolin, Class. Quantum Grav. **21**, 1725 (2004).
- [17] J. Kowalski-Glikman and L. Smolin, Phys. Rev. **D70**, 065020 (2004).